Journal of Engineering Physics and Thermophysics, Vol. 75, No. 4, 2002

MOTION OF A THIN OBLIQUE LAYER OF A PSEUDOPLASTIC FLUID

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UDC 532.501.32

Motion of a thin layer of the Williamson fluid over an inclined solid surface in the gravitational field is considered. Expressions are obtained for the stationary motion of the film in the gravitation field. The influence of the latter and tangential vibrations of the solid surface on the flow in the layer is analyzed. It is shown that the vibrations generate a pronounced averaged fluid flow even in the weak gravitational field in which the film is at rest in the absence of vibrations. The effect of the nonmonotonic dependence of the mean flow velocity of the film on the vibration frequency of the surface is found.

In the majority of works devoted to the motion of films of viscoplastic materials, for instance in [1–3], use is made of the discontinuous rheological Shvedov–Bingham model which makes it difficult to provide a unified description of the complicated nonstationary motion of the film in the entire region of the flow. In this case, it seems reasonable to use rheological models that preserve a physically important property of viscoplastic media, i.e., an abrupt decrease in flowability at small shear rates, and at the same time are continuous. Of particular interest are the rheological equations that allow a limiting transition to the Shvedov–Bingham model. These requirements are satisfied by the rheological Williamson model [4].

The present work is devoted to an investigation of the motion of a thin layer of the Williamson fluid over an inclined infinite solid surface performing translational vibrations in its plane. An analysis is made of the influence of the vibration frequency, amplitude, and the gravitational field on the flow in the layer.

Steady Flow. For convenience of comparison with the case of the absence of vibrations we will consider the steady flow of the Williamson fluid film over an immovable plane. Let a layer of the Williamson fluid of thickness h be restricted with a free surface on the one side and with a solid surface on another. Location of the coordinate axes and the angle of inclination of the layer α are shown in Fig. 1.

The components of the tensor of viscous tresses of the Williamson fluid are determined by the rheological relation [4]

$$\tau_{ij} = \left(\frac{A}{B + \sqrt{I_2}} + \mu_\infty \right) e_{ij} , \quad e_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad I_2 = \frac{1}{2} e_{ij} e_{ji} . \tag{1}$$

At $A = 0$ model (1) turns into the classical Newtonian one. At small values of the parameter B the Williamson fluid is close in its properties to the Bingham plastic [1, 2, 4]; in this case the rheological parameter A plays the role of the limiting shear stress.

With the aid of scale factors for space coordinates h, the fluid velocity $U^* = \rho g h^2 \cos \alpha / \mu_\infty$, stresses $\rho g h \cos \alpha$, and time $\rho h^2 / \mu_\infty$, we write the equation of motion of the fluid in the gravitational field in the dimensionless form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathrm{Re}\, (\mathbf{u} \nabla)\, \mathbf{u} = - \nabla p + \mathrm{Div}\, \tau + \mathbf{k} , \quad \mathrm{div}\, \mathbf{u} = 0 , \quad \tau_{ij} = \left(\frac{a}{b + \sqrt{I_2}} + 1 \right) e_{ij} . \tag{2}$$

We will consider the plane-parallel steady flow along the axis X at which all the quantities depend only on the coordinate z. In this case, Eqs. (2) and the boundary conditions on the solid wall and free surface are written as

[a]Perm' State University, Russia; email: lyubomov@psu.ru; [b]Perm' State Technical University; email: danata@freemail.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 75, No. 4, pp. 123–127, July–August, 2002. Original article submitted October 9, 2001.

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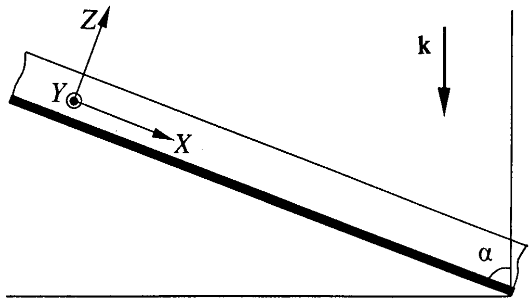


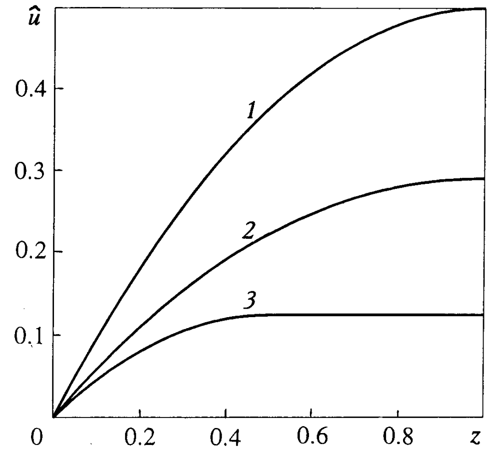


Fig. 1. Oblique layer of the Williamson fluid.

Fig. 2. Profiles of the basic flow for the cases: 1) Newtonian fluid, $a = 0$; 2) $a = 1$, $b = 1$; 3) $a = 0.5$, $b = 10^{-4}$.

$$\hat{\tau}_z = -1\,,\ \ \hat{p}_z = -\tan\alpha\,,\ \ \hat{\tau} = \left(\frac{a}{b + |\hat{u}_z|} + 1\right)\hat{u}_z\,,\ \ z = 0:\ \hat{u} = 0\,;\ \ z = 1:\ \hat{\tau} = 0\,,\ \hat{p} = 0\,. \tag{3}$$

After integration of system (3) for the transverse velocity that has the meaning of a shear rate and for the fluid velocity itself we obtain the expressions

$$\hat{u}_z = -\frac{1}{2}\left(a + b - (1 - z)\right) + \frac{1}{2} P(z)\,, \tag{4}$$

$$\hat{u} = -\frac{1}{2}\left(\frac{1}{2} z^2 + (a + b - 1)\, z\right) + \frac{1}{4} zP(z) + \frac{1}{4}(a - b - 1)\left(P(z) - P_0(z)\right) + ab \ln\left(\frac{|z + a - b - 1 + P(z)|}{|a - b - 1 + P_0(z)|}\right). \tag{5}$$

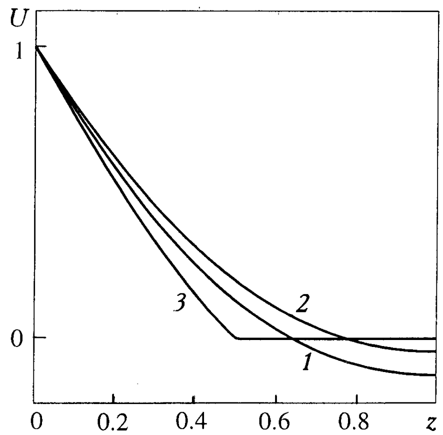
where $P(z) = \sqrt{(a + b - (1 - z))^2 + 4b(1 - z)}$ and $P_0(z) = \sqrt{(a + b - 1)^2 + 4b}$.

Figure 2 shows the velocity profiles of the basic flow for the values of the rheological parameters corresponding to different limiting cases. Curve 1 is constructed for the Newtonian fluid, i.e., at $a = 0$. In this case, from (5) for the velocity of the film flow we obtain $\hat{u} = -0.5z^2 + z$, which coincides with the expression given in [5]. Curve 2 constructed for $a = b = 1$ corresponds to the velocity of the film flow of the pseudoplastic fluid (1). The flow of the fluid of type (1) with the rheological parameters a and b comparable in value does not differ qualitatively from the Newtonian fluid flow. If $a \gg b$ and $b \to 0$, then, as seen from the form of curve 3 in Fig. 2 constructed for $a = 0.5$ and $b = 10^{-4}$, the fluid layer can be conventionally subdivided into liquid and quasisolid zones. Expression (4) for the shear rate at $b = 0$ has the form

$$\hat{u}_z = -\frac{1}{2}(a - 1 + z) + \frac{1}{2}\sqrt{(a - 1 + z)^2} = \begin{cases} 0\,, & z \ge 1 - a\,, \\ 1 - (z + a)\,, & z < 1 - a\,. \end{cases}$$

Within the limits of the quasisolid layer ($z \ge 1 - a$) the shear rates at small b are virtually equal to zero. The position of the interface of the zones is determined by the value of the parameter a. Henceforward the quasisolid layer will be referred to as the solid one.

Influence of Vibrations on the Fluid Flow. Let the solid surface perform oscillatory motion in its plane according to the law $U_0 \exp(i\omega t)$ + c.c. (c.c. is a complex-conjugate quantity). As earlier, we will consider the plane-parallel flow of a thin film of the pseudoplastic fluid. We assume that the system considered is in a sufficiently strong constant gravitational field capable of causing noticeable steady flow of the film with the velocity determined by expression (5). The wall performs vibrations whose rate is assumed to be small: $\varepsilon = U_0/U^* \ll 1$. In making the quanti-



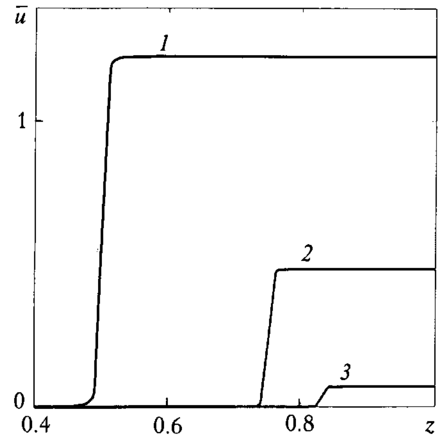


Fig. 3. Amplitudes of the unsteady contributions to the velocity of the basic flow at the vibration parameters $\Omega = 9$: 1, 2, and 3, notation is the same as in Fig. 2.

Fig. 4. Profiles of the steady contributions to the velocity of the basic flow: 1) $a = 0.5$, $\Omega = 9$; 2) 0.25 and 25; 3) 0.167 and 25.

ties dimensionless, we introduce a characteristic scale factor U_0 for velocity pulsations. In addition to the rheological parameters, in the problem the parameter Ω appears, which characterizes the vibration frequency.

Let us represent the velocity field and components of the tensor of viscous stresses in the form of a power series in terms of the small parameter ε:

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \dots , \quad \tau = \tau_0 + \varepsilon \tau_1 + \varepsilon^2 \tau_2 + \dots .$$

In this case, the main order of expansion of system (2) coincides with Eqs. (3) and determines the steady flow of the film.

In the next order, we obtain the equations and boundary conditions describing the pulsation flow of the fluid:

$$\tau_1 = F(z)\, u_{1z} , \quad u_{1t} = \tau_{1z} , \quad z = 0 : \quad u_1 = \exp(i\Omega t) + \text{c.c} , \quad z = 1 : \quad u_{1z} = 0 , \tag{6}$$

where $F(z) = 1 + ab/(b + \hat{u}_z)^2$.

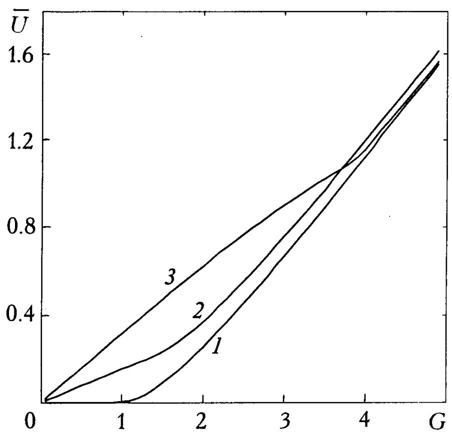
We will seek a solution of problem (6) in the form $u_1 = \mathrm{U}(z)\exp(i\Omega t) + \text{c.c.}$; then for the amplitude of unsteady contribution to the velocity of the steady flow we obtain

$$U_{zz} + \frac{F_z(z)}{F(z)} U_z - \frac{i\Omega}{F(z)} U = 0 , \quad z = 0 : \quad U = 1 , \quad z = 1 : \quad U_z = 0 . \tag{7}$$

Problem (7) was solved numerically by the Runge–Kutta–Merson method of fourth order. The profiles of the real part U for different values of the rheological parameters are presented in Fig. 3. Curves 1 and 2 are indicative of the qualitative similarity of the flows of the Newton and Williamson fluids if the rheological parameters of the latter are comparable in value to each other. The pulsation flow of the pseudoplastic fluid of the form of (1) at $a \gg b$ (Fig. 3, curve 3) is stratified into flow of the liquid zone adjacent to the wall and motion of the external solid zone.

Unlike the Newtonian fluid, vibrations applied to the Williamson fluid lead, because of nonlinearity of its rheological model, to generation of a median flow that enhances the steady motion. To elucidate the nature of the medium flow, we will write (2) in the following order of expansion in terms of ε. Omitting the indices and applying averaging to the equations obtained we have the following expression for the mean contribution to the velocity of the steady flow:

$$\bar{u}_z = 2 \frac{ab}{(b + \hat{u}_z)^3 + ab\,(b + \hat{u}_z)} \left| U_z \right|^2 , \quad z = 0 : \quad \bar{u} = 0 . \tag{8}$$



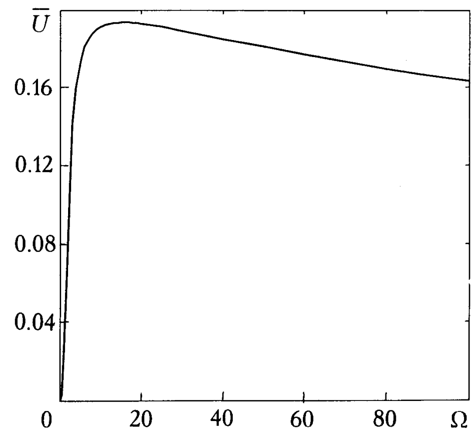


Fig. 5. Mean velocity of motion of the free film surface $\overline{U}$ versus the gravitational field: 1) $\Omega = 0$; 2) 900; 3) 9.

Fig. 6. Mean velocity of motion of the free film surface versus the dimensionless vibration frequency for the case $G = 0.6$.

Results of integration of (8) with account for (4) and (7) at $b = 10^{-4}$ are presented in Fig. 4. Of great interest is the case $a >> b$ and $b \to 0$ where the moving layer of the fluid is stratified into the liquid and solid zones. Within the limits of the liquid zone, the small-amplitude vibrations virtually do not exert an influence on the mean velocity of the fluid flow. At the same time, the external solid zone moves more quickly in the presence of vibrations than in their absence (the effect of "shaking" of the viscoplastic fluid from the vibrating surface [1]). At $a \sim b$, vibrations of the solid wall qualitatively do not change the mode of the averaged flow of the Williamson fluid film but make a non-zero contribution to the mean flow velocity over the entire thickness of the film.

To investigate the influence of the gravitational field on motion of the thin film of the viscoplastic fluid, it is convenient to use the ratios of the dimensionless parameters $b/a = B\mu_\infty/A$, $\varepsilon/a = U_0\,\mu_\infty/Ah$, and the parameter $G = 1/\varepsilon$ as the governing ones. It is seen that the first two do not depend on the gravitational force and the parameter G allows variation of the influence of the gravitational field on the film motion. Here, the scale of measuring the velocity of the fluid flow changes as

$$u\,(a, b, \varepsilon, \Omega) = au\,(b/a, \varepsilon/a, G, \Omega)\,. \tag{9}$$

Let us substitute (9) into system (2) and write it together with the boundary conditions for the case of the plane-parallel flow under the assumption that the parameter ε is a finite quantity:

$$u_t = \tau_z + G\,, \quad \tau = \left(\frac{1}{b/a + |\hat{u}_z|} + 1 \right) \hat{u}_z\,, \quad z = 0:\ \ u = \frac{\varepsilon}{a} \exp\,(i\Omega t) + \text{c.c}\,, \quad z = 1:\ \ u_z = 0\,. \tag{10}$$

Problem (10) was solved by the method of finite differences. It was assumed that $b/c = 10^{-3}$ and $\varepsilon/a = 1$. The initial approximation for the velocity distribution was prescribed by expressions (4) and (5) written with allowance for (9) where $\Omega = 0$. The dependence of the mean velocity of the fluid (on the free surface of the film) $\overline{U}$ on the gravitational field for different values of the parameter Ω is presented in Fig. 5. Curve 1 illustrates the case where vibrations are absent ($\Omega = 0$). Its form indicates that for $G < 1$ motion of the pseudoplastic fluid film is virtually absent. On applying vibrations (curves 2 and 3), a noticeable averaged motion of the fluid layer develops in the indicated range of the parameter G. With an increase in G, vibrations exert an ever-decreasing influence on the fluid motion and bring in only small corrections for the velocity value. The calculations for the parameters $G \geq 4$ and $\Omega = 9$ (such values of G and Ω correspond to the case of small ε) have shown that the influence of vibrations on the median flow of the fluid is noticeable only in the external solid layer, which agrees with the results obtained on solving problem (7).

Figure 6 illustrates the dependence of the mean velocity of motion of the free boundary of the film on the vibration frequency at $G = 0.6$. With an increase in the vibration frequency to $\Omega \approx 15$, an abrupt increase in the fluid velocity is observed, which is explained by the increase in acceleration acquired by the film on applying vibrations. With a further increase in the vibration frequency, the velocity of the free surface of the film decreases because the flow acquires the features of the boundary layer. Near the solid surface a viscous Stokes layer is formed in which the influence of vibrations is most pronounced. As Ω increases, the thickness of this layer decreases; consequently, the influence of vibrations on motion of the entire film becomes weaker.

Thus, the thin film of the Williamson fluid moving in the gravitational field over the inclined surface (1), in which the rheological parameter $B \to 0$, is stratified into the viscous flow zone and the quasisolid zone which corresponds to the mode of a viscoplastic fluid flow. Application of small-amplitude vibrations leads to an increase in the velocity of the averaged steady flow of the quasisolid zone, while the mean velocity of the fluid in the viscous zone remains unchanged and is determined only by the gravitational force. The influence of the gravitational field and vibration frequency on the flow of the viscoplastic fluid film has been investigated. It is shown that vibrations generate a noticeable averaged fluid flow even in weak gravitational fields in which the film is at rest on a solid surface in the absence of vibrations. The effect of the nonmonotonic dependence of the mean velocity of the flow film on the vibration frequency is found: for the prescribed rheological parameters such a vibration frequency exists at which the mean velocity of the flow is at its maximum. As the vibration frequency increases from zero to some threshold value, the fluid velocity abruptly increases, which is attributable to the increase in acceleration; the latter is acquired by the film on applying vibrations to it. The further increase in the vibration frequency leads to a decrease in the film velocity as a consequence of the fact that the flow acquires the features of the boundary layer.

NOTATION

h, thickness of the fluid layer, m; X, Y and Z, coordinate axes; α, angle of inclination of the layer; $\mathbf{k}$, unit vector along the direction of the gravitational force; g, free-fall acceleration, m/sec^2; τ_{ij}, components of the tensor of viscous stresses; e_{ij}, components of the tensor of shear rates; I_2, second invariant of the tensor of shear rates; A and B, rheological parameters of the fluid, respectively, kg/(m·sec^2) and sec^{-1}; μ_∞, dynamic viscosity of the fluid at large shear rates, Pa·sec; ρ, fluid density, kg/m^3; U^*, characteristic velocity of the fluid flow, m/sec; $\mathbf{u}$, dimensionless velocity vector; t, time, sec; $\mathrm{Re} = U^*\rho h/\mu_\infty$, Reynolds number; p and τ, dimensionless pressure and tensor of viscous stresses; $a = A/\rho gh \cos\alpha$, $b = B\mu_\infty/\rho gh \cos\alpha$, dimensionless rheological parameters; ∇, gradient; div, divergence of the vector field; Div, divergence of the tensor field; $\hat{u}$, $\hat{p}$, and $\hat{\tau}$, velocity, pressure, and tangential shear stress of the steady flow; U_0, amplitude of the vibration rate of the solid wall, m/sec; ω, vibration frequency, sec^{-1}; ε, dimensionless amplitude of the vibration rate; $\Omega = \omega\rho h^2/\mu_\infty$, dimensionless vibration frequency; u_i, τ_j, where $i = 0, ..., n$, coefficients of expansion of the velocity and components of the tensor of viscous stresses into a power series in terms of ε; U, dimensionless amplitude of the unsteady contribution to the velocity of the basic flow; $\bar{u}$, dimensionless mean contribution to the velocity of the basic steady flow; $G = a^{-1}$, dimensionless parameter determining the value of the gravitational field; $\overline{U}$, dimensionless mean velocity of the fluid motion over the free surface of the film. Subscripts: t, x, y, and z, derivatives with respect to time and coordinates.

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